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## LETTER TO THE EDITOR

## Leading off-diagonal contribution to the spectral form factor of chaotic quantum systems

Marko Turek and Klaus Richter

Institut für Theoretische Physik, Universität Regensburg, D-93040, Germany

E-mail: marko.turek@physik.uni-regensburg.de

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### Abstract

We semiclassically derive the leading off-diagonal correction to the spectral form factor of quantum systems with a chaotic classical counterpart. To this end we present a phase space generalization of a recent approach for uniformly hyperbolic systems (Sieber M and Richter K 2001 *Phys. Scr.* T **90** 128, Sieber M 2002 *J. Phys. A: Math. Gen.* **35** L613). Our results coincide with corresponding random matrix predictions. Furthermore we study the transition from the Gaussian orthogonal to the Gaussian unitary ensemble.

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Advanced semiclassical methods have been very successful in significantly improving our understanding of complex, classically chaotic quantum systems [3, 4]. This holds particularly true for observables which can be deduced from the (single-particle) Green function of the quantum system, such as the density of states, photo absorption, or orbital magnetism, to name a few. However, for quantities which are based on Green function products the situation is much more involved. These include linear response functions, e.g., for quantum transport, spectral correlation functions or, more generally, spectral statistics. A semiclassical treatment of such quantities is usually faced with the serious problem of evaluating multiple infinite sums over phase-carrying classical paths, which arise from the semiclassical representation of the Green functions in the limit  $\hbar \rightarrow 0$ . A prominent example is the spectral two-point correlator or its Fourier transform the spectral form factor  $K(\tau)$ . Random matrix assumptions leading to the prediction of a universal form for  $K(\tau)$  for classically chaotic quantum systems are supported by experimental and numerical data for a vast number of systems from different disciplines in physics [3]. Nevertheless, a corresponding analytical approach proving the conjectured [5] universality by explicitly including the underlying chaotic classical dynamics is still lacking. In this respect semiclassical techniques, bridging classical and quantum dynamics, appear to be natural tools but have to cope with the above-mentioned problems when evaluating spectral correlations. For the form factor this involves the computation of (energy) averaged double sums over periodic orbits which has been addressed in several semiclassical approaches [6].

Recently,  $K(\tau)$  was semiclassically investigated for uniformly hyperbolic two-dimensional systems where the classical dynamics is governed by a single Lyapunov exponent [1, 2]. By going beyond the usual diagonal approximation [7], ( $K^{(1)}(\tau) \approx 2\tau$ , describing the limit of spectral long-range correlations), the next to leading order contribution to  $K(\tau)$  was computed for systems with time-reversal symmetry. This was achieved by identifying off-diagonal pairs of correlated periodic orbits which are associated with each other via self-crossings in configuration space. Based on this orbit class the random matrix theory (RMT) prediction for the form factor in the Gaussian orthogonal ensemble (GOE),  $K(\tau) \approx 2\tau - 2\tau^2$  for  $\tau \rightarrow 0$ , could be derived [1, 2], see also [8].<sup>1</sup> However, the important question remains whether the above result, as well as the RMT prediction, is specific for systems with uniformly hyperbolic dynamics, or whether they pertain to the much broader class of chaotic systems with different periodic orbits having different Lyapunov exponents.

Here we present a generalization of the semiclassical approach outlined above to such non-uniformly hyperbolic systems in two dimensions and show that under rather general conditions the term  $\sim -2\tau^2$  in  $K(\tau)$  is indeed retained. To this end we develop a canonically invariant approach which is based on phase space arguments only. We identify as the relevant objects ‘crossing regions’ in phase space which can involve more than one self-crossing in configuration space. We express the action differences of the considered orbit pairs in terms of local phase space properties, the directions of the stable and unstable manifolds, and present a method for counting the ‘crossing regions’. This allows us eventually to determine their contribution to  $K(\tau)$ .

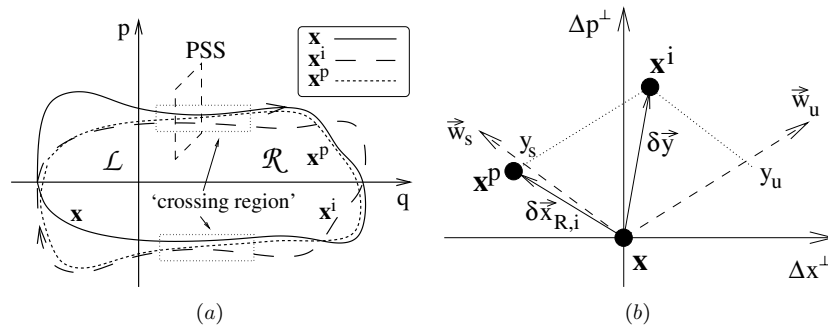
The semiclassical limit implies a large energy  $E$  compared to the mean level spacing  $1/\bar{d}(E)$ . The energy dependence of  $K(\tau)$  is smoothed out by an average over a classically small but quantum mechanically large energy window of size  $\Delta E$ . In the considered limit  $\Delta E$  can be chosen such that  $1/\bar{d}(E) \ll \Delta E \ll E$ . In this energy regime all classical actions, such as the action  $S_{\text{spo}}$  of the shortest periodic orbit, are much larger than  $\hbar$ , i.e.  $S_{\text{spo}}/\hbar \gg 1$ . Then one can employ Gutzwiller’s trace formula [3] for the oscillating part of the density of states and evaluate the Fourier transform of the spectral two-point correlation function. This gives for the form factor a double sum over periodic orbits [7]

$$K(\tau) = \frac{1}{T_H} \sum_{\gamma, \bar{\gamma}} \left\langle A_\gamma A_{\bar{\gamma}}^* \exp\left(i \frac{S_{\gamma, \bar{\gamma}}}{\hbar}\right) \delta\left(T - \frac{T_\gamma + T_{\bar{\gamma}}}{2}\right) \right\rangle_{\Delta E} \quad (1)$$

with the time scaled according to  $\tau = T/T_H$  where  $T_H = 2\pi\hbar\bar{d}(E)$  is the Heisenberg time. We represent each periodic orbit  $\gamma$  in terms of its phase space coordinates  $\mathbf{x} = (\mathbf{q}, \mathbf{p})$  where  $\mathbf{q}$  and  $\mathbf{p}$  are the position and momentum coordinates in two dimensions. In (1),  $T_\gamma$  is the period of an orbit  $\mathbf{x}_\gamma$  and  $A_\gamma$  includes both, its weight and Maslov index. The action difference between two orbits  $\mathbf{x}_\gamma$  and  $\mathbf{x}_{\bar{\gamma}}$  is given by  $S_{\gamma, \bar{\gamma}}$ . The argument of the double sum is a rapidly oscillating function and hence the energy average suppresses most of the terms. To obtain a nonvanishing contribution to  $K(\tau)$  the action difference must be small, i.e.  $S_{\gamma, \bar{\gamma}} \lesssim \hbar$ . Thus, for the time-reversal case, the largest contribution is due to pairs of one path  $\mathbf{x}_\gamma$  with itself or its time-reversed partner  $\mathbf{x}_\gamma^i = (\mathbf{q}_\gamma(T_\gamma - t), -\mathbf{p}_\gamma(T_\gamma - t))$  with vanishing action difference. Including only this type of pairs, known as the diagonal approximation [7], reproduces the linear contribution  $K^{(1)}(\tau) \approx 2\tau$ .

In the recent approach beyond the diagonal approximation pairs of closely related periodic orbits with a small action difference have been constructed by analysing self-crossings of the orbits in configuration space [1, 2]. Here we proceed in a different way by generalizing the approach via self-crossings to ‘crossing regions’ in phase space. We will show that if a

<sup>1</sup> The approach [1, 2] has been recently applied to quantum transport [8], to quantum graphs [9], to the Hadamard–Gutzwiller model [10] and to systems with non-conventional time reversal symmetry [11].



**Figure 1.** (a) Sketch of a correlated orbit pair in phase space (shown is a projection of the four-dimensional space). The original periodic orbit  $x$ , the time-reversed orbit  $x^i$  and the partner orbit  $x^p$  are represented by the solid, dashed and dotted lines, respectively. Due to time-reversal symmetry each ‘crossing region’ appears twice. The Poincaré surface of section (PSS) defined by the perpendicular coordinates  $(\Delta x^\perp, \Delta p^\perp)$  is indicated in the upper ‘crossing region’. (b) PSS within the ‘crossing region’ at the beginning of loop  $\mathcal{R}$ . The path  $x$  defines the centre of the coordinate system while the vector  $\delta \vec{y}$  points towards the time-reversed orbit  $x^i$ . The position of the partner orbit  $x^p$  is given by the vector  $\delta \vec{x}_{R,i}$ .

periodic orbit  $x_\gamma$  comes close to its time-reversed version  $x_\gamma^i$  in such a ‘crossing region’ it can imply the existence of yet another periodic orbit  $x_{\bar{\gamma}} = x_\gamma^p$  with a small action difference  $S_{\gamma,\bar{\gamma}}$  between the two. The partner orbit  $x_\gamma^p$  follows the original  $x_\gamma$  in a first segment  $\mathcal{R}$  and then the time-reversed path  $x_\gamma^i$  during the rest of the time in the second segment  $\mathcal{L}$ , see figure 1(a). It will turn out that not every ‘crossing region’ implies the existence of a partner orbit and therefore a small number of non-relevant ‘crossing regions’ has to be excluded.

To compute the contribution of the described orbit pairs  $(x_\gamma, x_\gamma^p)$  to  $K(\tau)$  we rearrange (1) into a sum over periodic orbits and another sum over all the partners with small action difference. This is based on the assumption that the dominant off-diagonal contribution to  $K(\tau)$ , equation (1), is due to the systematic correlation of actions of the considered orbit pairs  $(x_\gamma, x_\gamma^p)$  while other correlations are negligible. Then we sort the terms in the sums with respect to their action differences. Since the orbit length  $T_\gamma$  is proportional to  $T_H$  for fixed  $\tau$ , one expects a large number of ‘crossing regions’ for each orbit. This allows us to replace the sum over action differences  $S_{\gamma,\bar{\gamma}}$  by an integral. The first off-diagonal contribution to  $K(\tau)$  then reads

$$K^{(2)}(\tau) = 4\tau \operatorname{Re} \left\langle \int_0^\infty dS \left\langle \frac{dN_{S,\gamma}}{dS} \right\rangle_{(\gamma,T)} \exp\left(i\frac{S}{\hbar}\right) \right\rangle_{\Delta E} \quad (2)$$

where  $N_{S,\gamma}$  is the number of relevant ‘crossing regions’ for a given periodic orbit  $x_\gamma$  with an associated action difference smaller than  $S$ . In equation (2),  $\langle \dots \rangle_{(\gamma,T)}$  denotes a weighted average over all orbits  $x_\gamma$  of given length  $T_\gamma = \tau T_H$ . It is defined as

$$\left\langle \frac{dN_{S,\gamma}}{dS} \right\rangle_{(\gamma,T)} \equiv \frac{1}{T} \sum_\gamma \frac{dN_{S,\gamma}}{dS} |A_\gamma|^2 \delta(T - T_\gamma). \quad (3)$$

We proceed with the evaluation of (2) by first determining the geometry of the partner orbit  $x_\gamma^p$ . Then we show that the action difference between  $x_\gamma$  and  $x_\gamma^p$  is indeed small if the orbit  $x_\gamma$  and its time-reversed version  $x_\gamma^i$  come close together in parts of the phase space. Finally we derive

the averaged number  $\langle dN_{S,\gamma}/dS \rangle_{(\gamma, T_\gamma=T)}$  of relevant ‘crossing regions’ in the contributing regime  $S \lesssim \hbar$ .

*Partner geometry.* To construct the partner orbit we analyse the linearized equations of motion around  $x_\gamma$  in part  $\mathcal{R}$  and around  $x_\gamma^i$  in section  $\mathcal{L}$  in the Poincaré surface of section (PSS), see figure 1, and show that a nontrivial solution representing the partner orbit  $x_\gamma^p$  exists under certain conditions. The distance between the original orbit  $x_\gamma$  and its time-reversed partner  $x_\gamma^i$  in the PSS defined by the local transverse coordinates at the phase space position  $x \equiv x_\gamma(t)$  is given by the vector  $\delta\vec{y} \equiv (\Delta x^\perp, \Delta p^\perp)$ .<sup>2</sup> In general, the time evolution of a small deviation is determined by the starting point  $x_0$  of the original path in phase space and the initial deviation in the PSS  $\delta\vec{y}_0$ , e.g.  $\delta\vec{y} = \delta\vec{y}(t; x_0, \delta\vec{y}_0)$ . Let us assume for the moment that the ‘crossing region’ under consideration is characterized by a small distance  $\delta\vec{y}$ . Decomposing  $\delta\vec{y} = y_u \vec{w}_u + y_s \vec{w}_s$  in terms of the local unstable and stable manifolds  $\vec{w}_{u,s}$  [12] with the expansion coefficients

$$y_{u,s} \equiv \frac{\vec{w}_{s,u}^T Z \delta\vec{y}}{\vec{w}_{s,u}^T Z \vec{w}_{u,s}} \quad \text{and} \quad Z \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4)$$

the smallness of  $\delta\vec{y}$  is given if  $|y_{u,s}| \ll 1$ . For simplicity we choose the relative orientation and lengths of the  $\vec{w}_{u,s}$  such that  $(Z\vec{w}_s)^T \vec{w}_u = S_{\text{spo}}$ . It will turn out that  $|y_{u,s}| \ll 1$  is the relevant regime for the evaluation of  $K(\tau)$  in the semiclassical limit. The distance between  $x_\gamma$  and the partner orbit  $x_\gamma^p$  at the beginning of the first loop  $\mathcal{R}$  is denoted by  $\delta\vec{x}_{R,i}$ , see figure 1(b). This vector lies in the PSS defined at the phase space position  $x \equiv x_\gamma(t)$  before the loop  $\mathcal{R}$  (it corresponds to the upper crossing region in figure 1(a)). Having passed loop  $\mathcal{R}$  after time  $T_R$  this distance has changed to  $R\delta\vec{x}_{R,i}$  with  $R$  being the stability matrix for loop  $\mathcal{R}$ . Before (and after) the other part of the orbit the difference between the time-reversed path  $x_\gamma^i$  and the partner  $x_\gamma^p$  is denoted by  $\delta\vec{x}_{L^i,i}$  (and  $L^i \delta\vec{x}_{L^i,i}$ ) where  $L^i \equiv FL^{-1}F$  is the stability matrix of the time-reversed loop  $\mathcal{L}$ . The matrix  $F$  is defined as

$$F \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

Solving the linearized equations of motion under the condition that the two parts of the partner orbit fit together in the ‘crossing regions’ yields the geometry of  $x_\gamma^p$  in terms of the distance  $\delta\vec{y}$  between the original orbit  $x_\gamma$  and its time-reversed  $x_\gamma^i$ :

$$\delta\vec{x}_{R,i} = [1 - L^i R]^{-1} [1 - L^i F] \delta\vec{y}. \quad (6)$$

The corresponding condition for  $\delta\vec{x}_{L^i,i}$  for the lower ‘crossing region’ in figure 1(a) is found in a similar way [15]. This set of solutions  $\delta\vec{x}$  includes terms of order  $\mathcal{O}(y_{u,s})$  and defines the partner orbit for a given small  $\delta\vec{y}$  representing a ‘crossing region’.

We now argue that a ‘crossing region’ does not yield a partner if the periodic orbit  $x_\gamma$  lies close to a self-retracing path during one of the loops. This type of ‘crossing region’ is described by a  $\delta\vec{y}^{sr}$  such that the original path  $x_\gamma$  and the time-reversed  $x_\gamma^i$  stay close together with  $|y_{u,s}^{sr}| \ll 1$  holding during the entire loop  $\mathcal{R}$ . The motion of the time-reversed path  $x_\gamma^i$  in  $\mathcal{R}$  can then be obtained by linearization around the original  $x_\gamma$  using the stability matrix  $R$ . In this case one finds  $R\delta\vec{y}^{sr} = F\delta\vec{y}^{sr}$  neglecting corrections of higher than first order in  $y_{s,u}$ . The solution (6) is then replaced by  $\delta\vec{x}_{R,i} = \delta\vec{y}^{sr}$  and  $\delta\vec{x}_{L^i,i} = 0$ . It, therefore, does not give a new partner orbit but just the time-reversed periodic orbit  $x_\gamma^p = x_\gamma^i$ . But contributions to  $K(\tau)$  of this type are already treated in the diagonal approximation and must not be included in (2). However,

<sup>2</sup> For clarity of notation we write vectors in the PSS using an arrow (e.g.,  $\delta\vec{y}$ ) in contrast to vectors in phase space (e.g.,  $x$ ).

because of the hyperbolic nature of the dynamics the condition  $|y_{u,s}^{sr}| \ll 1$  holds true for the entire loop  $\mathcal{R}$  only if the loop time  $T_R^{sr}$  is smaller than a certain minimal time  $T_{R,\min}(\mathbf{x}_\gamma(t), \delta\vec{y})$ , i.e.  $T_R^{sr} < T_{R,\min}$ . Corresponding arguments involving the time  $T_{L,\min}(\mathbf{x}_\gamma(t), \delta\vec{y})$  hold for the other loop  $\mathcal{L}$ . These minimal times  $T_{(R,L),\min}$  are determined by the time scale on which the linearization breaks down and are implicitly given by the condition

$$y_{u,s}(\pm T_{(R,L),\min}; \mathbf{x}, \delta\vec{y}) = c_{u,s}(\mathbf{x}) \quad (7)$$

where the ‘+’ (‘-’) sign corresponds to  $y_u$  ( $y_s$ ) and  $c_{u,s}(\mathbf{x})$  are constants of order one.

Since (6) uniquely defines the partner orbit as sketched in figure 1(a) one can show that the Maslov index for the orbit  $\mathbf{x}_\gamma$  equals that of  $\mathbf{x}_\gamma^p$ . The Maslov index of a periodic orbit is given by the winding number of the stable or unstable manifold [14]. Since the partner orbit is close to the original orbit in section  $\mathcal{R}$  the contribution to the winding number accumulated between the two ‘crossing regions’ is the same for both. The second contribution comes from loop  $\mathcal{L}$  and is thus given by the geometry  $\mathbf{x}_\gamma^i$ . The total winding number for  $\mathbf{x}^p$ , the sum of these two contributions, can be related to the total winding number of  $\mathbf{x}$  in the following way. Time reversal symmetry implies the relation  $\vec{w}_{s,u}(\mathbf{x}_\gamma(t)) = -F\vec{w}_{u,s}(\mathbf{x}_\gamma^i(T_\gamma - t))$  between the manifolds of  $\mathbf{x}_\gamma$  and  $\mathbf{x}_\gamma^i$ . With this relation one can show that the contributions to the winding number coming from  $\mathbf{x}_\gamma^i$  and  $\mathbf{x}_\gamma$  during  $\mathcal{L}$  are equal. Then the equality of the Maslov indices of  $\mathbf{x}_\gamma$  and  $\mathbf{x}_\gamma^p$  becomes evident [16] if one uses similar arguments as in the proof of the equality of the Maslov indices of a periodic orbit and its time-reversed counterpart [15].

*Action difference.* The geometry of the partner given by (6) allows one to derive the action difference between the two orbits of the pair  $(\mathbf{x}_\gamma, \mathbf{x}_\gamma^p)$  as a function of  $\delta\vec{y}$ . Since the distance  $\delta\vec{y}$  is assumed to be small it is sufficient to expand the action in  $\delta\vec{y}$ . However, the expression one obtains for  $S$  [17] still contains all the elements of the stability matrices  $R$  and  $L$  because the geometry of the partner orbit as given by (6) depends on them. Since the existence of the partner implies loop lengths larger than  $T_{(L,R),\min}$  the vectors  $\delta\vec{x}$  have to lie very close to the respective local stable or unstable manifolds, e.g.,  $\delta\vec{x}_{R,i} \approx y_s\vec{w}_s(\mathbf{x})$ . This fact enables us to express the action difference  $S_{\gamma,\bar{\gamma}}$  between the original orbit  $\mathbf{x}_\gamma$  and the partner orbit  $\mathbf{x}_\gamma^p = \mathbf{x}_{\bar{\gamma}}$  in terms of the local manifolds [16] and the expansion coefficients  $y_{u,s}$  given by (4). Under the assumption that the directions of the manifolds are continuous functions of the position in phase space [12] the result then reads

$$S_{\gamma,\bar{\gamma}} = S(\delta\vec{y}, \mathbf{x}) \approx (\vec{w}_u^T Z \vec{w}_s) y_s y_u \quad (8)$$

which is correct up to second order in  $\delta\vec{y}$ . In the semiclassical limit it is sufficient to consider the regime given by  $|y_{u,s}| \sim \sqrt{\hbar / (\vec{w}_u^T Z \vec{w}_s)} = \sqrt{\hbar / S_{\text{spo}}} \ll 1$  which justifies the above restriction to small values of  $|y_{u,s}|$ .

Equations (6) representing the geometry of  $\mathbf{x}^p$  and (8) are invariant under a shift of the PSS along the orbit within a ‘crossing region’. This also implies that a ‘crossing region’ may include several self-crossings in configuration space and hence the number of partners is not necessarily given by the number of self-crossings as was the case for the uniformly hyperbolic systems [1, 2].

*Counting ‘crossing regions’.* Moving the PSS along  $\mathbf{x}_\gamma$  each ‘crossing region’ is characterized by a  $\delta\vec{y}$  that starts close to  $\vec{w}_s$  and ends almost parallel to  $\vec{w}_u$ . The number of ‘crossing regions’ can thus be determined by counting how often the unstable components  $y_u$  of the vectors  $\delta\vec{y}$  go through a certain fixed value  $y_u^c$  as one moves along  $\mathbf{x}_\gamma$ . This parameter  $y_u^c$  fixes the position within the ‘crossing regions’ used to identify and count it. The total number of ‘crossing regions’ must not depend on  $y_u^c$  and it will be shown that this is indeed the case.

To evaluate the weighted average (3) of the number  $N_S$  of contributing ‘crossing regions’ we make use of the following sum rule [13] valid for ergodic systems:

$$\frac{1}{T} \sum_{\gamma} |A_{\gamma}|^2 \delta(T - T_{\gamma}) \int_0^{T_{\gamma}} dt f(\mathbf{x}_{\gamma}(t)) \approx \int_0^T dt f(\mathbf{x}(t)) \quad \text{for } T \rightarrow \infty. \quad (9)$$

It relates the weighted average of a function  $f(\mathbf{x})$  over all periodic orbits of length  $T$  to a time average over a generic ergodic trajectory  $\mathbf{x}(t, \mathbf{x}_0)$  starting at any  $\mathbf{x}_0$  in phase space. To apply (9) in the calculation of (3) one writes the number of events where the time-reversed path comes close to the original one as

$$\frac{d^2 N_{S,\gamma}(\mathbf{x}_{\gamma}(t))}{dS dt} = \rho(\mathbf{x}_{\gamma}(t), S, y_u^c) \frac{\dot{y}_u(0; \mathbf{x}_{\gamma}(t), S, y_u^c)}{y_u(0; \mathbf{x}_{\gamma}(t), S, y_u^c)} \quad (10)$$

where  $\rho(\mathbf{x}_{\gamma}(t); S, y_u^c)$  is the density of partners per action in the PSS located at  $\mathbf{x}_{\gamma}(t)$ . The ratio  $\dot{y}_u/y_u$  describes the velocity of the flow in the PSS so that (10) indeed gives the number of partners per action and time. In expression (10) the position in the PSS is specified in terms of  $S$  and  $y_u^c$  using (8).

The density  $\rho(\mathbf{x}; S, y_u^c)$  is determined by the probability that the time-reversed path goes through the point  $(S, y_u^c)$  in the PSS defined at  $\mathbf{x}$ . The long time limit is thus given by the ergodic density  $\rho_0 = T/\Sigma(E) = \tau/(2\pi\hbar)$  with  $\Sigma(E) = (2\pi\hbar)^2 \bar{d}(E)$  being the volume of the energy surface in phase space. However, since certain ‘crossing regions’ with short loop lengths characterized by  $T_{(R,L)} < T_{(R,L),\min}$ , see (7), do not yield a partner one has to exclude parts of the time-reversed path of length

$$T_{\min}(\mathbf{x}, S) \equiv 2[T_{R,\min}(\mathbf{x}, S, y_u^c) + T_{L,\min}(\mathbf{x}, S, y_u^c)] \quad (11)$$

with the factor 2 coming from time reversal symmetry. According to definition (7) this time  $T_{\min}(\mathbf{x}, S)$  is determined by the time it takes for the unstable component  $y_u$  to grow from the small value  $S/(S_{\text{spo}} c_s) \ll 1$  to the value  $c_u \sim 1$ . It therefore does not depend on  $y_u^c$ . To compute (3) we first apply (9) to (10). Then the minimal time (11) is of the same order as the Ehrenfest time and in the regime of small action differences  $S \sim \hbar$  given by the asymptotic expression

$$\lambda T_{\min}(\mathbf{x}, S) \approx 2 \ln \left[ c_u(\mathbf{x}) c_s(\mathbf{x}) \frac{S_{\text{spo}}}{S} \right] \gg 1 \quad (12)$$

where we used the standard definition of the Lyapunov exponent  $\lambda$  in terms of any long ergodic path [12]. Using this relation we find that the main contribution to (2) in the semiclassical limit is given by the averaged number of ‘crossing regions’

$$\left\langle \frac{dN_{S,\gamma}}{dS} \right\rangle_{\gamma} = \rho_0 \left[ \lambda T + 2 \ln \frac{S}{S_{\text{spo}}} - \text{const} \right] \quad (13)$$

where const depends on the structure of the phase space of the considered system but is independent of the action difference  $S$  and the time  $T$  [16]. The first of the three contributions on the rhs of (13) is the largest one ( $\sim \hbar^{-2}$ ) and represents the ergodic properties of the system. The second term  $\sim \hbar^{-1} \ln \hbar$  reflects the underlying dynamics of the system<sup>3</sup> and is much smaller than the first but still logarithmically larger than the third term  $\sim \hbar^{-1}$ . Inserting (13) into (2) one finds that it is this logarithmic correction to the ergodic behaviour that gives

<sup>3</sup> The crucial logarithmic dependence has been numerically confirmed for the cardioid billiard [16].

the RMT result

$$K^{(2)}(\tau) \approx -2\tau^2. \quad (14)$$

*GOE–GUE transition.* The crossover between the universality classes as time reversal symmetry is broken has been originally semiclassically obtained in [18] within the diagonal approximation. Here we summarize a dynamical evaluation of this transition for the first off-diagonal correction. The appropriate transition parameter  $\alpha$  interpolating between GOE and GUE is given by the ratio between the root mean square of a typical time-reversal symmetry breaking matrix element and the mean level spacing [18]. We shall consider the case where the symmetry is broken by a uniform magnetic field  $B$  perpendicular to a uniformly hyperbolic two-dimensional system. Since for an orbit pair  $(x_\gamma, x_\gamma^p)$  one of the common loops ( $\mathcal{L}$  or  $\mathcal{R}$ ) between the ‘crossing regions’ is traversed in the opposite direction in configuration space, the orbit pair acquires, owing to the overall magnetic flux enclosed, an additional action difference  $4\pi\mathcal{A}B/\phi_0$ . Here  $\mathcal{A}$  is the enclosed (directed) area of the loop and  $\phi_0 = h/2e$  the flux quantum. For hyperbolic systems the distribution of directed areas enclosed by trajectories of length  $t$  is, to good approximation [4], Gaussian with variance  $t\beta$  where  $\beta$  is a system specific parameter. To compute  $K^{(2)}(\tau)$  at finite  $B$  we must additionally integrate, for given loop length  $t$ , the flux-induced action differences over the Gaussian area distribution. This results in a further damping  $\exp[-t/t_B]$  with  $1/t_B = 2\beta(2\pi B/\phi_0)^2$ . Counting the ‘crossing regions’ with this additional weight eventually gives, together with the diagonal term [16],<sup>4</sup>

$$K(\tau; \alpha) \approx \tau [1 + (1 - 2\tau) e^{-8\pi^2\alpha^2\tau}] \quad \text{for } \tau \rightarrow 0 \quad (15)$$

with  $\alpha^2\tau = (B/\phi_0)^2\beta T$ . This precisely coincides with the form factor of parametric RMT [20] in the short time limit.

To conclude, we have shown how correlations in the action of classical paths determine the spectral statistics of the quantum mechanical energy eigenvalues. We derived the next to leading order contribution for the spectral form factor  $K(\tau)$  and showed that it is identical to the corresponding RMT result. Our derivation is canonical invariant and based on phase space arguments only. This leads to the result that correlations in the classical action in hyperbolic chaotic systems are caused by ‘crossing regions’ in phase space where an orbit and its time-reversed version come close together. Since our method avoids the concept of crossings in configuration space it is suited to be extended to systems with more than two degrees of freedom.

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<sup>4</sup> This result has also recently been obtained for a quantum graph in a weak magnetic field by Nagao and Saito [19].

<sup>5</sup> After we finished this paper we received a preprint by D Spehner which contains some similar material.



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